

# Heavy-Tailed Innovations in the R Package `stochvol`

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## Abstract

We document how sampling from a conditional Student's  $t$  distribution is implemented in `stochvol`. Moreover, a simple example using EUR/CHF exchange rates illustrates how to use the augmented sampler. We conclude with results and implications.

*Keywords:* Student's  $t$  distribution, data augmentation, EUR/CHF exchange rates.

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## Preface

This note serves as a preliminary add-on to the more elaborate article “Dealing with Stochastic Volatility in Time Series using the R package `stochvol`” (Kastner 2016a). It discusses and relaxes the restriction to conditionally normal errors in the vanilla stochastic volatility (SV) model.

## 1. The SV model with Student's $t$ errors

Several authors have suggested to use non-normal conditional innovation distributions for stochastic volatility modeling. Examples include the Student's  $t$  distribution (Harvey, Ruiz, and Shephard 1994), the extended Generalized Inverse Gaussian (Silva, Lopes, and Migon 2006), (semi-)parametric innovations (Jensen and Maheu 2010; Delatola and Griffin 2011), or the GH skew Student's  $t$  distribution (Nakajima and Omori 2012). In the following, we describe how the estimation of the SV model with Student's  $t$  errors is implemented in the R (R Core Team 2016) package `stochvol` (Kastner 2016b).

Let  $\mathbf{y} = (y_1, y_2, \dots, y_n)^\top$  be a vector of returns with mean zero. The SV model with Student's  $t$  errors (in short SV- $t$ ) is given through

$$y_t | h_t, \nu \sim t_\nu(0, \exp h_t), \quad (1)$$

$$h_t | h_{t-1}, \mu, \phi, \sigma_\eta \sim \mathcal{N}(\mu + \phi(h_{t-1} - \mu), \sigma_\eta^2), \quad (2)$$

$$h_0 | \mu, \phi, \sigma_\eta \sim \mathcal{N}(\mu, \sigma_\eta^2 / (1 - \phi^2)), \quad (3)$$

i.e., conditionally on  $h_t$ , the data is assumed to follow a zero-mean non-standardized Student's  $t$  distribution with  $\nu$  degrees of freedom and variance  $(\nu \exp h_t) / (\nu - 2)$  for  $\nu > 2$ . Following Chib, Nardari, and Shephard (2002), we assume that a priori the degrees of freedom parameter

$\nu \sim \mathcal{U}(a, b)$ , i.e., follows a uniform distribution with support on the real interval  $(a, b)$ . All other prior components are chosen as in [Kastner \(2016a\)](#).

## 2. Usage

Estimating a stochastic volatility model with conditional  $t$  errors via **stochvol** is very similar to estimating a model with standard Gaussian errors, differing only through specifying a non-NA argument `priornu`. This triggers the sampler specified in Section 3. To provide an example, we investigate the historical daily EUR/CHF exchange rates and display these in Figure 1.

```
R> library(stochvol)
R> data(exrates)
R> par(mfrow = c(2, 1), mar = c(1.7, 1.7, 1.7, 0.1), mgp = c(1.6, 0.6, 0))
R> plot(exrates$date, exrates$CHF, type = 'l', main = 'Price of 1 EUR in CHF')
R> dat <- logret(exrates$CHF, demean = TRUE)
R> plot(exrates$date[-1], dat, type = 'l', main = 'Demeaned log returns')
```

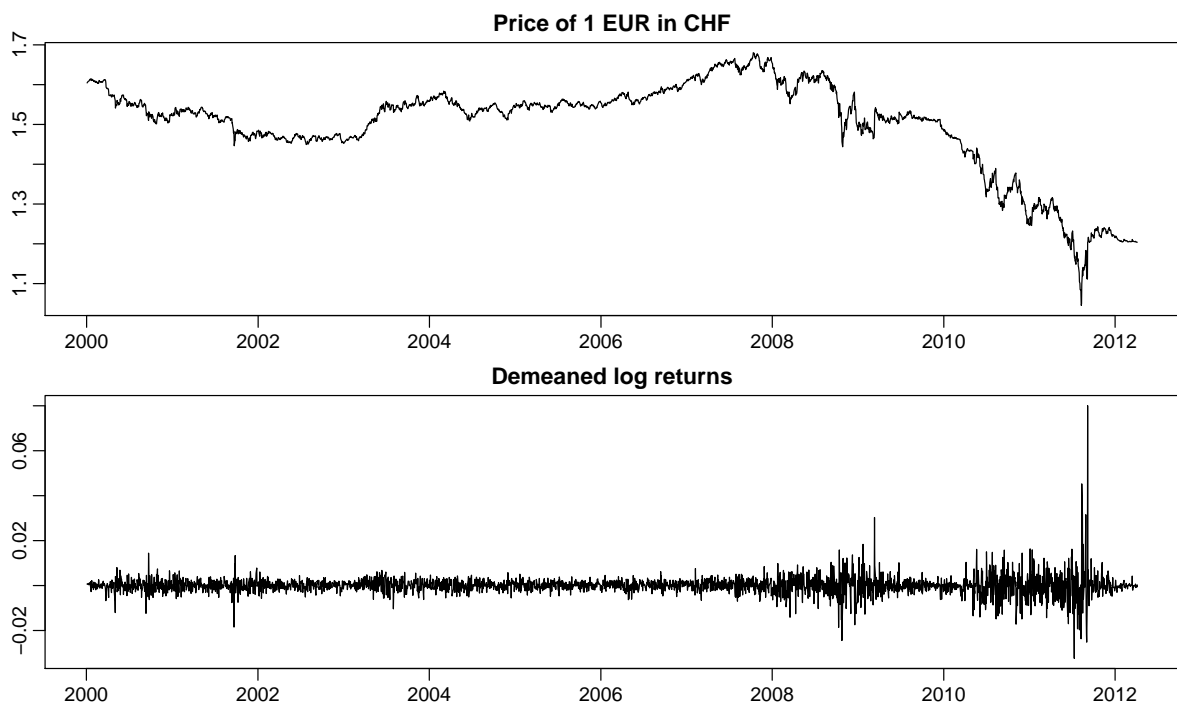


Figure 1: Levels and demeaned log returns of EUR in CHF.

By specifying the argument `priornu` (a two-element vector containing the lower and upper bounds of the uniform prior for  $\nu$ ), we can trigger the sampler to allow for heavy-tailed conditional innovations.

```
R> rest <- svsample(dat, priormu = c(-12, 1), priorphi = c(20, 1.1),
+   priorsigma = 0.1, priornu = c(2, 100), burnin = 2000)
R> plot(rest, showobs = FALSE)
```

Results are displayed in Figure 2, containing the output from the SV- $t$  model. Row 1 depicts  $\exp(h_t/2)$  for  $t \in \{1, \dots, n\}$ ; row 2 shows the time varying standard deviations given through  $\sqrt{\nu/(\nu-2)} \exp(h_t/2)$  for  $t \in \{1, \dots, n\}$ ; row 3 portrays trace plots and row 4 outlines the corresponding smoothed kernel density estimates for the four parameters  $\mu$ ,  $\phi$ ,  $\sigma$ , and  $\nu$ . It is worth noting that  $\nu$  is estimated to lie between 6 and 18 with high posterior probability, indicating evidence for the presence of heavy tails even after catering for stochastic volatility. The extra flexibility of the SV- $t$  sampler seems to allow for increased persistence  $\phi$  and smaller variance of log-volatility  $\sigma^2$ , resulting in smoother time-varying volatility estimates.

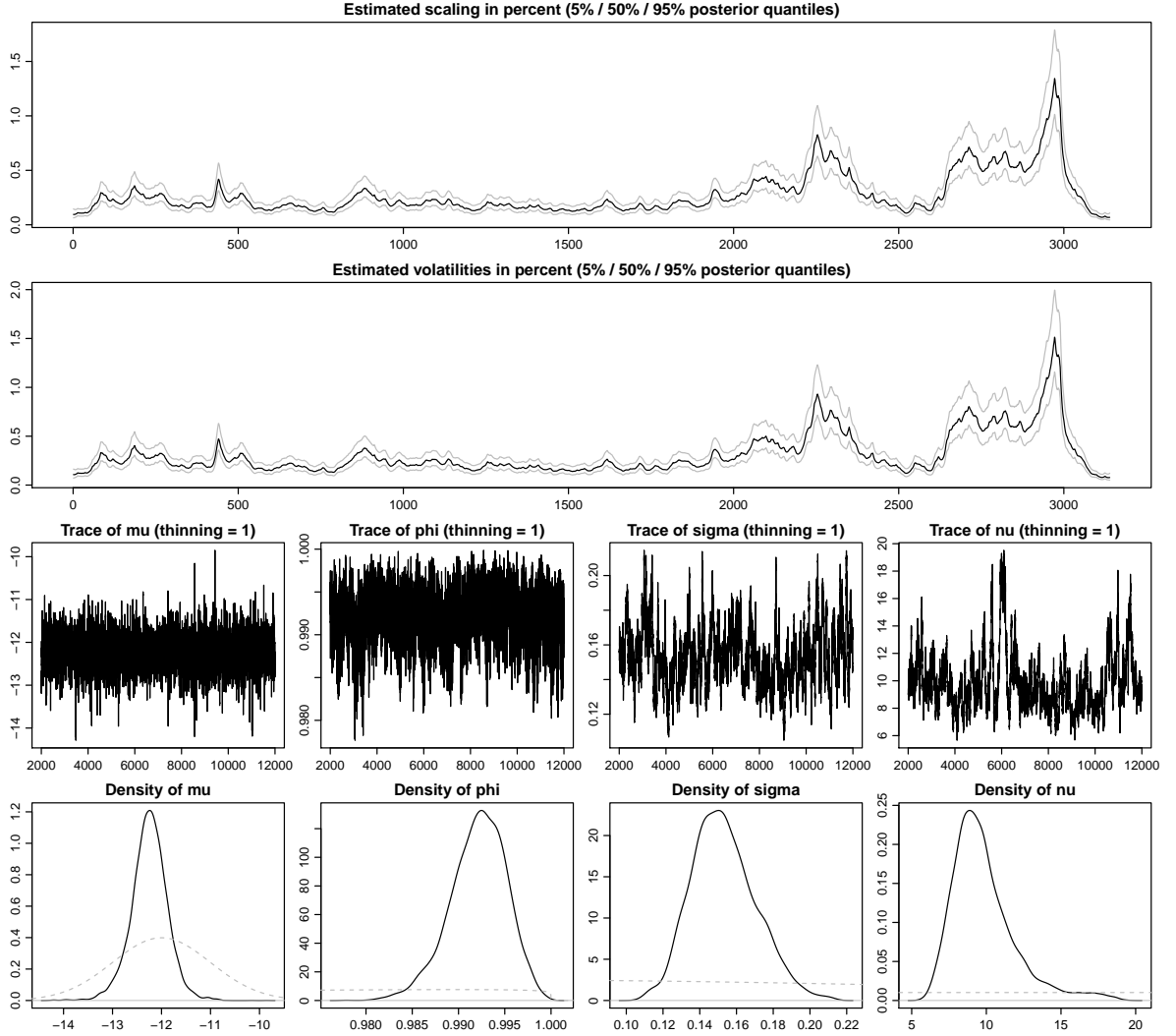


Figure 2: Standard output of the `plot` method when applied to an `svdraws` object containing posterior draws from an SV model with Student's  $t$  errors.

We investigate one-day-ahead out-of-sample predictive performance of an AR(1) model with (a) homoskedastic, (b) SV, (c) SV- $t$ , (d) GARCH(1,1) errors, applied to the raw exchange rate data. Details about this procedure are provided in Chapter 5 of [Kastner \(2016a\)](#). The results, summarized in Figure 3, speak in favor of the SV- $t$  model for this dataset.

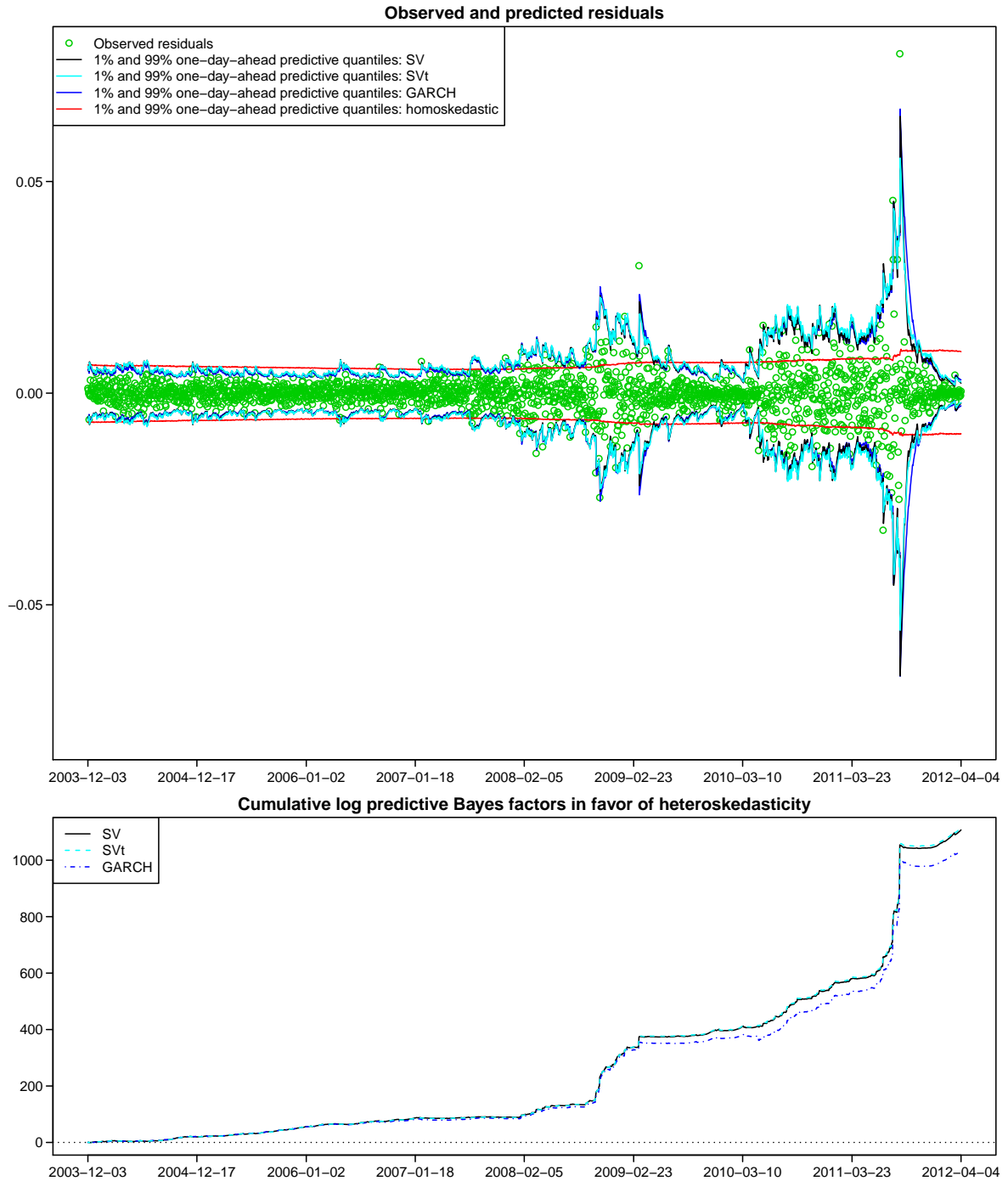


Figure 3: Log predictive one-day-ahead Bayes factors in favor of SV, SV- $t$ , and GARCH errors over the homoskedastic model. The final log predictive Bayes factors aggregate to 1107.44 (SV), 1115.36 (SV- $t$ ), and 1033.53 (GARCH), respectively, thus providing strong evidence for the SV- $t$  model.

### 3. Necessary modifications in the sampling scheme

The Student's  $t$  distribution appearing in Equation 1 can be conveniently expressed as a scale mixture of normal distributions,

$$\begin{aligned} y_t | h_t, \tau_t &\sim \mathcal{N}(0, \tau_t \exp h_t), \\ \tau_t | \nu &\sim \mathcal{G}^{-1}(\nu/2, \nu/2), \end{aligned}$$

where  $\mathcal{N}(\mu, \sigma_\eta^2)$  denotes the normal distribution with mean  $\mu$  and variance  $\sigma_\eta^2$ , and  $\mathcal{G}^{-1}(a, b)$  denotes the inverse gamma distribution with shape and scale parameters  $a$  and  $b$ , respectively.

Treating  $\boldsymbol{\tau} = (\tau_1, \dots, \tau_n)^\top$  as latent data and letting  $\tilde{y}_t = y_t / \sqrt{\tau_t}$  for  $t \in \{1, \dots, n\}$ , we have

$$\tilde{y}_t | h_t \sim \mathcal{N}(0, \exp h_t),$$

and the AWOL sampler described in [Kastner and Frühwirth-Schnatter \(2014\)](#) can directly be applied to the transformed data. To obtain draws from the newly introduced variables  $\boldsymbol{\tau}$  and  $\nu$ , two additional steps are required.

#### 3.1. Sampling the auxiliary variables

It is easy to see that the marginal posterior is given through

$$\tau_t | y_t, h_t, \nu \sim \mathcal{G}^{-1}\left(\frac{\nu + 1}{2}, \frac{\nu + y_t^2 \exp(-h_t)}{2}\right),$$

independently for each  $t \in \{1, \dots, n\}$ . Obtaining draws from this distribution is straightforward.

#### 3.2. Sampling the degrees of freedom parameter

The full conditional posterior of the degrees of freedom parameter,  $\nu | \cdot$ , only depends on  $\boldsymbol{\tau}$ . Its density is given through the product of  $n$  univariate truncated inverse gamma densities which may be written as

$$p(\nu | \cdot) = p(\nu | \boldsymbol{\tau}) \propto \left(\frac{\nu}{2}\right)^{n\nu/2} \Gamma\left(\frac{\nu}{2}\right)^{-n} \left(\prod_{t=1}^n \tau_t\right)^{-\nu/2} \exp\left\{-\frac{\nu}{2} \sum_{t=1}^n \frac{1}{\tau_t}\right\} \quad (4)$$

for  $\nu \in (a, b)$  and zero elsewhere.

For obtaining draws from this distribution, we use an independence Metropolis-Hastings update. We follow [Chib and Greenberg \(1994\)](#), who introduced the idea of specifying an independence proposal through numerical maximization of the log-density. For the problem at hand, we consequently aim for optimizing

$$\log p(\nu | \boldsymbol{\tau}) = \frac{n\nu}{2} \log(\nu/2) - n \log \Gamma(\nu/2) - \frac{\nu}{2} \sum_{t=1}^n \left(\log \tau_t + \frac{1}{\tau_t}\right) + C, \quad (5)$$

with first and second derivatives given through

$$\frac{\partial \log p(\nu | \boldsymbol{\tau})}{\partial \nu} = \frac{n}{2} \left(1 + \log(\nu/2) - \psi^{(0)}(\nu/2)\right) - \frac{1}{2} \sum_{t=1}^n \left(\log \tau_t + \frac{1}{\tau_t}\right), \quad (6)$$

$$\frac{\partial^2 \log p(\nu | \boldsymbol{\tau})}{\partial \nu^2} = \frac{n}{2\nu} - \frac{n}{4} \psi^{(1)}(\nu/2), \quad (7)$$

where  $\psi^{(m)}$  denotes the polygamma function of order  $m$ . Using the above, it is easy to numerically find

$$\begin{aligned}\hat{\nu} &= \arg \max_{\nu} \log p(\nu|\boldsymbol{\tau}), \\ B_{\hat{\nu}} &= -1 \left/ \frac{\partial^2 \log p(\nu|\boldsymbol{\tau})}{\partial \nu^2} \right|_{\nu=\hat{\nu}},\end{aligned}$$

and a proposal candidate  $\nu_{\text{prop}}$  may be drawn from a normal distribution with mean  $\hat{\nu}$  and variance  $B_{\hat{\nu}}$  (the Laplace approximation). Letting  $\phi(x|\hat{\nu}, B_{\hat{\nu}})$  denote the corresponding density function, the acceptance probability is equal to  $\min\{1, R\}$  with

$$R = \frac{p(\nu_{\text{prop}}|\boldsymbol{\tau})}{p(\nu_{\text{old}}|\boldsymbol{\tau})} \times \frac{\phi(\nu_{\text{old}}|\hat{\nu}, B_{\hat{\nu}})}{\phi(\nu_{\text{prop}}|\hat{\nu}, B_{\hat{\nu}})}.$$

## 4. Conclusion

We have shown how a simply data augmentation trick can be utilized to generalize the core sampler in *stochvol* in order to cater for potentially heavier-tailed innovation distributions. However, several caveats are called for:

- Even though the uniform prior for  $\nu$  has been used widely, more robust alternatives are probably preferred, cf. [Frühwirth-Schnatter and Pyne \(2010\)](#) and the references therein.
- Sampling the degrees of freedom parameter  $\nu$  conditionally on  $\boldsymbol{\tau}$  can be very inefficient if  $\nu$  becomes large (and thus the plain vanilla SV model suffices). We recommend to resort to the original SV sampler in this case.
- Leaving aside the additional computational burden, it is trivial to incorporate this extension into samplers employing *stochvol* as part of a larger MCMC scheme (e.g. [Huber 2014](#); [Kastner, Frühwirth-Schnatter, and Lopes 2014](#); [Dovern, Feldkircher, and Huber 2015](#)). Nevertheless, at the current stage of development, this should be conducted with caution by carefully investigating the convergence of the posterior draws.

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