

# Solving the $N$ -Queens Problem with Local Search

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This vignette provides example code for a combinatorial problem: the *N-Queens Problem*.

## 1 The problem

The goal is to place  $N$  queens on a chess-board of size  $N \times N$  in such a way that no queen is attacked. A queen may move vertically, horizontally and on a diagonal. So whenever there is more than one queen on any row, column or diagonal, the position is invalid. To solve the problem with a Local Search (LS), we need three components:

1. a way to represent a solution (i.e. a position on the chessboard);
2. a way to evaluate such a solution;
3. and, since we use a LS, a method to modify a solution.

We start by attaching the package and fixing a seed.

```
> library("NMOF")  
> set.seed(134577)
```

## 2 Representing a solution

Since on any row there cannot be more than one queen, we may store a position as a vector of columns on which the queens are placed. (In chess, rows would be called ranks and columns would be files, but we prefer matrix terminology.) Thus, a candidate solution  $p$  ( $p$  for position) could look as follows:

```
> N <- 8          ## board size  
> p <- sample.int(N) ## a random solution  
> data.frame(row = 1:N, column = p)
```

|   | row | column |
|---|-----|--------|
| 1 | 1   | 1      |
| 2 | 2   | 7      |
| 3 | 3   | 2      |
| 4 | 4   | 4      |
| 5 | 5   | 3      |
| 6 | 6   | 8      |
| 7 | 7   | 5      |
| 8 | 8   | 6      |

Or (a very bad solution):

```
> p <- rep(1, N)  
> data.frame(row = 1:N, column = p)
```

|   | row | column |
|---|-----|--------|
| 1 | 1   | 1      |
| 2 | 2   | 1      |
| 3 | 3   | 1      |
| 4 | 4   | 1      |
| 5 | 5   | 1      |
| 6 | 6   | 1      |
| 7 | 7   | 1      |
| 8 | 8   | 1      |

We will also want to visualise a position, for which we write the function `print_board`.

```
> print_board <- function(p, q.char = "Q", sep = " ") {
  n <- length(p)
  row <- rep("-", n)
  for (i in seq_len(n)) {
    row_i <- row
    row_i[p[i]] <- q.char

    cat(paste(row_i, collapse = sep))
    cat("\n")
  }
}
> print_board(p)
```

```
Q - - - - -
Q - - - - -
Q - - - - -
Q - - - - -
Q - - - - -
Q - - - - -
Q - - - - -
Q - - - - -
```

### 3 Evaluating a solution

We need to compute on what row, column, diagonal (top left to bottom right) or reverse diagonal (top right to bottom left) a queen stands. Rows and columns are simple; we label the diagonals as follows.

```
> mat <- array(NA, dim = c(N,N)) ## diagonals
> for (r in 1:N)
  for (c in 1:N)
    mat[r,c] <- c - r
> mat
```

|      | [,1] | [,2] | [,3] | [,4] | [,5] | [,6] | [,7] | [,8] |
|------|------|------|------|------|------|------|------|------|
| [1,] | 0    | 1    | 2    | 3    | 4    | 5    | 6    | 7    |
| [2,] | -1   | 0    | 1    | 2    | 3    | 4    | 5    | 6    |
| [3,] | -2   | -1   | 0    | 1    | 2    | 3    | 4    | 5    |
| [4,] | -3   | -2   | -1   | 0    | 1    | 2    | 3    | 4    |
| [5,] | -4   | -3   | -2   | -1   | 0    | 1    | 2    | 3    |
| [6,] | -5   | -4   | -3   | -2   | -1   | 0    | 1    | 2    |
| [7,] | -6   | -5   | -4   | -3   | -2   | -1   | 0    | 1    |
| [8,] | -7   | -6   | -5   | -4   | -3   | -2   | -1   | 0    |

```
> mat <- array(NA, dim = c(N,N)) ## reverse diagonals
> for (r in 1:N)
  for (c in 1:N)
    mat[r,c] <- c + r - (N + 1)
> mat
```

|      | [,1] | [,2] | [,3] | [,4] | [,5] | [,6] | [,7] | [,8] |
|------|------|------|------|------|------|------|------|------|
| [1,] | -7   | -6   | -5   | -4   | -3   | -2   | -1   | 0    |
| [2,] | -6   | -5   | -4   | -3   | -2   | -1   | 0    | 1    |
| [3,] | -5   | -4   | -3   | -2   | -1   | 0    | 1    | 2    |
| [4,] | -4   | -3   | -2   | -1   | 0    | 1    | 2    | 3    |
| [5,] | -3   | -2   | -1   | 0    | 1    | 2    | 3    | 4    |
| [6,] | -2   | -1   | 0    | 1    | 2    | 3    | 4    | 5    |
| [7,] | -1   | 0    | 1    | 2    | 3    | 4    | 5    | 6    |
| [8,] | 0    | 1    | 2    | 3    | 4    | 5    | 6    | 7    |

Note that for reverse diagonals, the  $N + 1$  would not be necessary; it serves only to shift the diagonal labels so that the main diagonal is zero.

Thus for a given solution  $p$ , we know the row, column, diagonal and reverse diagonal for each queen. We define the quality of a solution by the number of attacks that happen: for a valid solution, that number should be zero.

```
> n_attacks <- function(p) {
  ## more than one Q on a column?
  sum(duplicated(p)) +

  ## more than one Q on a diagonal?
  sum(duplicated(p - seq_along(p))) +

  ## more than one Q on a reverse diagonal?
  sum(duplicated(p + seq_along(p)))
}
> n_attacks(p)
[1] 7
```

## 4 Changing a solution

A given position may be modified by picking one row randomly and then moving the queen there to the left or right. We allow for moves up to  $\text{step}$  squares, which we set to 3 in the example.

```
> neighbour <- function(p) {
  step <- 3
  i <- sample.int(N, 1)
  p[i] <- p[i] + sample(c(1:step, -(1:step)), 1)

  if (p[i] > N)
    p[i] <- 1
  else if (p[i] < 1)
    p[i] <- N
  p
}
```

```
> print_board(p)
```

```
Q - - - - -
Q - - - - -
Q - - - - -
Q - - - - -
Q - - - - -
Q - - - - -
Q - - - - -
Q - - - - -
```

```
> print_board(p <- neighbour(p))
```

```
- - - - - Q
Q - - - - -
Q - - - - -
Q - - - - -
Q - - - - -
Q - - - - -
Q - - - - -
Q - - - - -
```

```
> print_board(p <- neighbour(p))
```

```
- - - - - Q - -
Q - - - - - - -
Q - - - - - - -
Q - - - - - - -
Q - - - - - - -
Q - - - - - - -
Q - - - - - - -
Q - - - - - - -
Q - - - - - - -
```

## 5 Solving the model

We use three different LS methods: a ‘classical’ Stochastic Local Search (LSopt), Threshold Accepting (TAopt) and Simulated Annealing (SAopt).

```
> p0 <- rep(1, N) ## or a random initial solution: p0 <- sample.int(N)
> print_board(p0)
```

```
Q - - - - - - -
Q - - - - - - -
Q - - - - - - -
Q - - - - - - -
Q - - - - - - -
Q - - - - - - -
Q - - - - - - -
Q - - - - - - -
Q - - - - - - -
```

```
> sol <- LSopt(n_attacks, list(x0 = p0,
                              neighbour = neighbour,
                              printBar = FALSE,
                              nS = 10000))
```

```
Local Search.
Initial solution: 7
Finished.
Best solution overall: 1
```

```
> print_board(sol$xbest)
```

```
- - Q - - - - -
- - - - - Q - -
- - - - - - - Q
Q - - - - - - -
- - - Q - - - - -
- - - - - - Q -
- - - - Q - - - -
- - - - - - - Q
```

```
> sol <- TAopt(n_attacks, list(x0 = p0,
                              neighbour = neighbour,
                              printBar = FALSE,
                              nS = 1000))
```

```
Threshold Accepting

Computing thresholds ... OK
Estimated remaining running time: 0.195 secs

Running Threshold Accepting ...
Initial solution: 7
```

```
Finished.  
Best solution overall: 0
```

```
> print_board(sol$xbest)
```

```
- - - - - Q -  
- Q - - - - -  
- - - - - Q - -  
- - Q - - - - -  
Q - - - - -  
- - - Q - - - -  
- - - - - Q  
- - - - Q - - -
```

```
> sol <- SAopt(n_attacks, list(x0 = p0,  
                               neighbour = neighbour,  
                               printBar = FALSE,  
                               nS = 1000))
```

```
Simulated Annealing.  
  
Calibrating acceptance criterion ... OK  
Estimated remaining running time: 0.2 secs.  
  
Running Simulated Annealing ...  
Initial solution: 7  
Finished.  
Best solution overall: 0
```

```
> print_board(sol$xbest)
```

```
- - - - - Q - -  
- Q - - - - -  
- - - - - Q -  
Q - - - - -  
- - Q - - - - -  
- - - Q - - - -  
- - - - - Q  
- - - Q - - - -
```

## References

Manfred Gilli, Dietmar Maringer, and Enrico Schumann. *Numerical Methods and Optimization in Finance*. Elsevier/Academic Press, 2011. URL <http://enricoschumann.net/NMOF>.